New normal forms for proofs via deep inference

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Untangling cut-elimination

- In traditional cut-elimination procedures in Gentzen theory, we eliminate cut instances from proofs by moving upwards instances of the mix rule.

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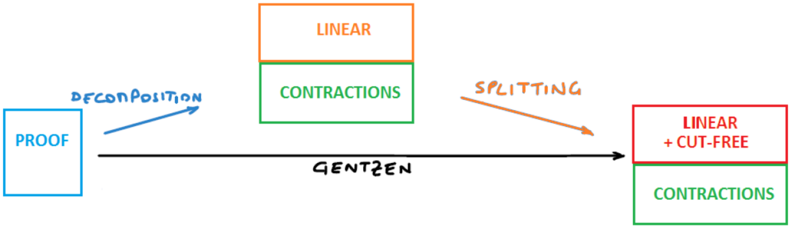
- The presence of contraction makes for a jump to a higher complexity class.
- Can we untangle cut and contraction and normalise on each of them separately and in a natural way?

Untangling cut- elimination



- Decomposition is the normalisation of contractions by permuting them to the bottom of proofs. It can increase the size of proofs exponentially.
- Splitting deals with cut-elimination in contraction-free systems. It does not generate meaningful complexity.

What is Deep Inference?

It's the free composition of derivations with the same connectives as formulae.

If

$$\phi = \begin{array}{c} A \\ \| \\ B \end{array} \quad \text{and} \quad \psi = \begin{array}{c} C \\ \| \\ D \end{array}$$

are two derivations, then

$$(\phi \vee \psi) = \begin{array}{c} A \\ \| \\ B \end{array} \vee \begin{array}{c} C \\ \| \\ D \end{array} \quad \text{and} \quad (\phi \wedge \psi) = \begin{array}{c} A \\ \| \\ B \end{array} \wedge \begin{array}{c} C \\ \| \\ D \end{array}$$

are valid derivations.

Why Deep Inference?

- To obtain new notions of normalisation in addition to cut elimination [7, 6].
- To get proof systems whose inference rules are local and highly regular [9].
- To express logics that cannot be expressed in Gentzen [11, 2].

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- To obtain new notions of normalisation in addition to cut elimination [7, 6].
- To get proof systems whose inference rules are local and highly regular [9].
- To express logics that cannot be expressed in Gentzen [11, 2].
- To shorten analytic proofs by exponential factors compared to Gentzen [4, 5].
- To inspire a new generation of proof nets and semantics of proofs [10].

Some proof systems

$$\mathsf{ai}\!\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \; \mathsf{ai}\!\uparrow \qquad \frac{a \wedge \bar{a}}{\mathsf{f}}$$

$$\mathsf{s}\, \frac{(A \vee B) \wedge C}{(A \wedge C) \vee B} \qquad \mathsf{m}\, \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\mathsf{ac}\!\downarrow \frac{a \vee a}{a} \qquad \mathsf{ac}\!\uparrow \frac{a}{a \wedge a}$$

$$\mathsf{aw}\!\downarrow \frac{\mathsf{f}}{a} \qquad \mathsf{aw}\!\uparrow \frac{a}{\mathsf{t}}$$

Figure: System SKS [3]

Some proof systems

$$\frac{1}{a \parr \bar{a}} \qquad \frac{a \otimes \bar{a}}{\bot}$$

$$\otimes\!\downarrow \frac{(A \parr B) \otimes (C \parr D)}{(A \otimes C) \parr (B \parr D)} \qquad \parr\!\uparrow \frac{(A \parr B) \otimes (C \otimes D)}{(A \otimes C) \parr (B \otimes D)}$$

$$\&\!\downarrow \frac{(A \parr B) \,\&\, (C \parr D)}{(A \,\&\, C) \parr (B \oplus D)} \qquad \oplus\!\uparrow \frac{(A \oplus B) \otimes (C \,\&\, D)}{(A \otimes C) \oplus (B \otimes D)}$$

$$\oplus\!\downarrow \frac{(A \parr B) \oplus (C \parr D)}{(A \oplus C) \parr (B \oplus D)} \qquad \&\!\uparrow \frac{(A \,\&\, B) \otimes (C \,\&\, D)}{(A \otimes C) \,\&\, (B \otimes D)}$$

$$\mathsf{m} \frac{(A \,\&\, B) \oplus (C \,\&\, D)}{(A \oplus C) \,\&\, (B \oplus D)}$$

$$\mathsf{m}_1 \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)} \qquad \mathsf{m}_2 \frac{(A \,\&\, B) \parr (C \,\&\, D)}{(A \parr C) \,\&\, (B \parr D)}$$

$$\mathsf{ac}\!\downarrow \frac{a \oplus a}{a} \qquad \mathsf{ac}\!\uparrow \frac{a}{a \,\&\, a}$$

$$\mathsf{aw}\!\downarrow \frac{0}{a} \qquad \mathsf{aw}\!\uparrow \frac{a}{\top}$$

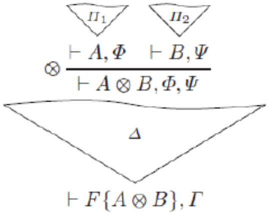
Figure: System SLLS [9]

Splitting

- Generalisation of a common technique employed for cut-elimination in deep inference systems.
- We split the proof in different pieces, and put them back together in such a way that we avoid the cut.
- This type of argument has been used to prove the admissibility of rules other than the atomic cut.

Splitting

- We find all the subproofs that are independent from each other above the multiplicative ‘cut’ connective. We will show that we can put them back together like a puzzle and obtain a proof with the same conclusion but without the cuts.



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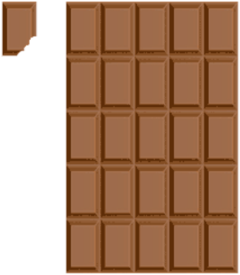

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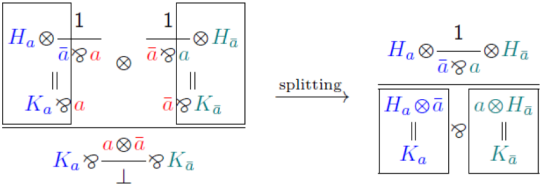

Splitting

Definition

A system S is *splittable* if:

1. There are dual distinguished connectives $\times$ with unit 1 and $+$ with unit 0.
2. S is uniquely composed of the rules

$$\text{ai}\!\downarrow \frac{1}{a + \bar{a}} \quad \text{and} \quad \text{ai}\!\uparrow \frac{a \times \bar{a}}{0} \quad ,$$

   together with rules

$$\alpha\!\downarrow \frac{(A + B)\ \alpha\ (C + D)}{(A\ \alpha\ C) + (B\ \check{\alpha}\ D)} \quad \text{and} \quad \alpha\!\uparrow \frac{(A\ \widehat{\alpha}\ B) \times (C\ \alpha\ D)}{(A \times C)\ \alpha\ (B \times D)}$$

   for every connective α.
3. For every unit u, $u + \bar{u} = 1$.
4. For every connective α, $1\ \widehat{\alpha}\ 1 = 1$.

Splitting

- This defines a whole class of substructural logics.
- It includes logics that support self-dual non commutative connectives, such as BV.
- It includes MLL.
- It includes the 'splittable fragment' of CL and MALL, i.e. the one made-up of all the rules that do not stem from atomic contraction.

Splitting

Theorem

Let S *be a splittable system. For every proof* $\begin{array}{c}\phi \,\Vert\, \mathsf{S} \\ A\end{array}$ *there is a proof* $\begin{array}{c}\psi \,\Vert\, \mathsf{S}\setminus\{\mathsf{ai}\uparrow, \alpha\uparrow\} \\ A\end{array}$ *linear on the size of* ϕ*, and where* ψ *can be obtained from* ϕ *in a procedure of polynomial-time complexity.*

Decomposition

- In several deep inference systems, we know that we can permute atomic contractions to the bottom of proofs through local reductions.

Decomposition

- In several deep inference systems, we know that we can permute atomic contractions to the bottom of proofs through local reductions.
- We want to permute general contractions

$$\frac{A \vee A}{A} \quad .$$

- It is essential to move away from the sequent calculus: it is always possible to build a valid sequent for which there is no sequent calculus proof where all the contractions are confined to the bottom [1].

Decomposition

- Reduction rule 1:

$$\mathsf{s}\,\frac{\dfrac{(A_1 \vee A_2) \vee (A_1 \vee A_2)}{A_1 \vee A_2} \wedge C}{(A_1 \wedge C) \vee A_2} \longrightarrow \frac{\mathsf{s}\,\dfrac{((A_1 \vee A_2) \vee (A_1 \vee A_2)) \wedge \dfrac{C}{(C \wedge C)}}{\mathsf{s}\,\dfrac{(A_1 \vee A_2) \wedge C}{(A_1 \wedge C) \vee A_2} \vee \mathsf{s}\,\dfrac{(A_1 \vee A_2) \wedge C}{(A_1 \wedge C) \vee A_2}}}{(A_1 \wedge C) \vee A_2}$$

- We create a cocontraction: locality is indispensable.

Decomposition

- Reduction rule 1:

$$\mathsf{ai}{\uparrow}\,\dfrac{\dfrac{a \vee a}{a} \wedge \bar{a}}{\mathsf{f}} \longrightarrow \mathsf{s}\,\dfrac{(a \vee a) \wedge \dfrac{\bar{a}}{(\bar{a} \wedge \bar{a})}}{\mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}}$$

- This is how we can permute contractions past cuts.

Decomposition

- Reduction rule 2:

$$\cfrac{\cfrac{A \vee A}{A}}{A \wedge A} \longrightarrow \cfrac{\cfrac{A}{(A \wedge A)} \vee \cfrac{A}{(A \wedge A)}}{A \wedge A}$$

- This can cause an exponential explosion in the size of the proof.

Decomposition

Definition

A system SD is *decomposable* if:

1. There are dual distinguished relations $\sqcup$ with unit w and $\sqcap$ with unit $\bar{w}$.
2. SD is composed of a splittable system S with, together with the rules

$$\frac{A \sqcup A}{A} \quad \text{and} \quad \frac{A}{A \sqcap A} \quad ,$$

$$\text{aw}{\downarrow}\,\frac{w}{a} \quad \text{and} \quad \text{aw}{\uparrow}\,\frac{a}{\bar{w}} \quad .$$

3. For every unit u, $u \sqcup u = u = u \sqcap u$.
4. For every connective α, $w \;\alpha\; w = w$ and $\bar{w} \;\alpha\; \bar{w} = \bar{w}$.

Decomposition

- The definition includes CL and MALL.
- It will be expanded to include exponentials.

Decomposition

Theorem

In a decomposable system, there is a reduction strategy so that every proof can be rewritten as a proof where all instances of contraction are at the bottom of the proof (and there are no instances of cocontraction).

Conclusions

- We can control complexity by following atoms.
- We give a uniform treatment for many existing logics
- We can use these results to design systems with guaranteed mosular cut-elimination.

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