Decomposition and cycles: isolating two complexity-generating mechanisms.

Andrea Aler Tubella

University of Bath

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Decomposition

- In many systems, derivations can be arranged into consecutive subderivations made up of only certain rules: we call this transformation decomposition.
- Herbrand's Theorem is an example: bottom phase with contraction and quantifier rules and a top phase with propositional rules only.

Decomposition

- We can achieve a particular decomposition result for classical logic by doing local rewritings of proofs.
- We “move” atomic contractions downwards in a proof, and cocontractions upwards [3].
- The procedure can easily be visualised graphically.

Decomposition

$$\mathsf{ai}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\,\frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\,\frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\,\frac{S\{a\}}{S(a,a)}$$

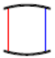
Figure: SKS [1]

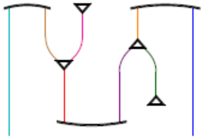
Atomic Flows [3]

$$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\boxed{\dfrac{\mathsf{t}}{a \vee \bar{a}}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\boxed{\mathsf{s}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \mathsf{t}}} \vee \mathsf{t}}$$

$$= \dfrac{\boxed{= \dfrac{\boxed{\dfrac{\mathsf{t}}{\bar{a} \vee a}} \vee \boxed{\dfrac{\mathsf{f}}{a}}}{\bar{a} \vee \boxed{\dfrac{a \vee a}{a}}}} \wedge \boxed{\dfrac{\mathsf{t}}{\boxed{\dfrac{\bar{a}}{\bar{a} \wedge \boxed{\dfrac{\bar{a}}{\mathsf{t}}}}} \vee a}}}{\mathsf{s}^2\,\dfrac{[\bar{a} \vee a] \wedge [\bar{a} \vee a]}{\bar{a} \vee \boxed{\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee a}}$$

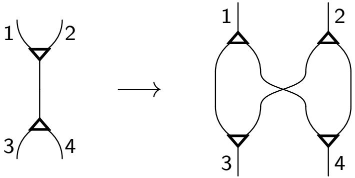
$$\boxed{\mathsf{m}\,\dfrac{\boxed{\dfrac{a}{a \wedge a}} \vee \boxed{\dfrac{b}{b \wedge b}}}{[a \vee b] \wedge [a \vee b]}} \wedge \boxed{\dfrac{a}{a \wedge a}}$$





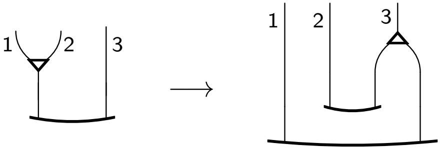
Decomposition

$$
\begin{array}{c}
ac{\downarrow}\,\dfrac{a^1 \vee a^2}{\;a\;} \\
ac{\uparrow}\,\dfrac{}{a^3 \wedge a^4}
\end{array}
\quad\longrightarrow\quad
m\,\dfrac{\boxed{ac{\uparrow}\,\dfrac{a^1}{a \wedge a}} \vee \boxed{ac{\uparrow}\,\dfrac{a^2}{a \wedge a}}}{\boxed{ac{\downarrow}\,\dfrac{a \vee a}{a^3}} \wedge \boxed{ac{\downarrow}\,\dfrac{a \vee a}{a^4}}}
$$



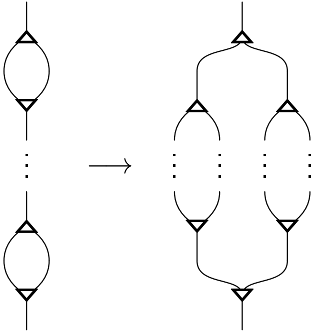
Decomposition

$$
\text{ai}\!\uparrow \frac{\boxed{\text{ac}\!\downarrow \dfrac{a^1 \vee a^2}{a}} \wedge \bar{a}^3}{\mathsf{f}}
\quad \longrightarrow \quad
= \frac{s\,\dfrac{s\,\dfrac{(a^1 \vee a^2) \wedge \boxed{\text{ac}\!\uparrow \dfrac{\bar{a}^3}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{\text{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{\text{ai}\!\uparrow \dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}}
$$



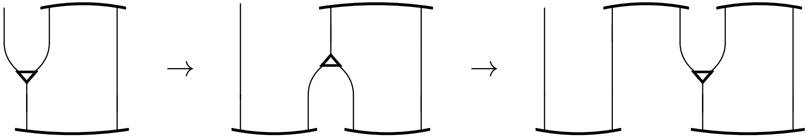
Complexity

- The decomposition procedure may increase the size of a proof exponentially.



Cycles

- When we apply the reductions to atomic contractions that belong to a cycle, the rewriting system is not terminating:



- Cycles come from the connexion of an introduction and a cut.

Why this decomposition?

- Not only decomposing proofs, but generally derivations.
- Separation of compression mechanisms
  - By separating into a phase with cuts and a phase with contractions, we divide cut-elimination into two separate procedures.
- Easily represented graphically.
- Seemingly more general than classical logic.
  - Analogous local reductions can be defined for LL [4].

Not all is settled

- Cycles.
  - Independence of decomposition from cut-elimination.
  - Proof compression? [2]
- Full decomposition into linear/non-linear phases.
- Generality.

A new methodology, that we call subatomic, allows us to tackle all three questions.

One shape to rule them all

- Many proof systems can be represented in such a way that every inference rule is an instance of a single linear inference rule scheme.

$$\frac{(A \; \alpha \; B) \; \beta \; (C \; \alpha' \; D)}{(A \; \beta \; C) \; \alpha \; (B \; \beta' \; D)}$$

But how?

- This shape arises very often when we have atomic introduction and contraction rules.

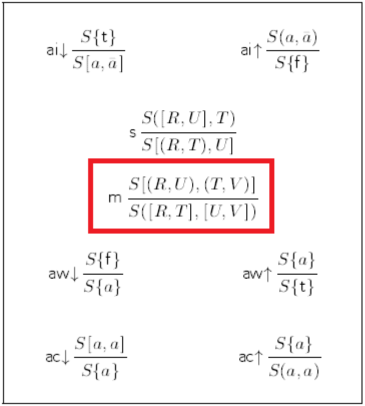


Figure: SKS [1]

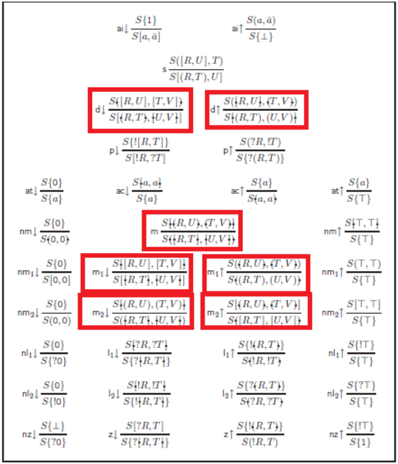


Figure: SLLS [4]

But how?

- How do the atomic rules fit the scheme?

We can consider atoms as superpositions of truth values:

$$\mathsf{f}\ a\ \mathsf{t} \mapsto a \quad \mathsf{t}\ a\ \mathsf{f} \mapsto \bar{a}$$
$$\mathsf{t}\ a\ \mathsf{t} \mapsto \mathsf{t} \quad \mathsf{f}\ a\ \mathsf{f} \mapsto \mathsf{f}$$

- How does that change the rules?

Contraction:

$$\frac{(A\ a\ B) \vee (C\ a\ D)}{(A \vee C)\ a\ (B \vee D)}$$

But how?

- How do the atomic rules fit the scheme?

We can consider atoms as superpositions of truth values:

$$\mathsf{f}\ a\ \mathsf{t} \mapsto a \quad \mathsf{t}\ a\ \mathsf{f} \mapsto \bar{a}$$
$$\mathsf{t}\ a\ \mathsf{t} \mapsto \mathsf{t} \quad \mathsf{f}\ a\ \mathsf{f} \mapsto \mathsf{f}$$

- How does that change the rules?

Contraction:

$$\frac{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{t})}{(\mathsf{f} \vee \mathsf{f})\ a\ (\mathsf{t} \vee \mathsf{t})} \mapsto \frac{a \vee a}{a}$$

But how?

- How do the atomic rules fit the scheme?

We can consider atoms as superpositions of truth values:

$$\mathsf{f}\ a\ \mathsf{t} \mapsto a \quad \mathsf{t}\ a\ \mathsf{f} \mapsto \bar{a}$$
$$\mathsf{t}\ a\ \mathsf{t} \mapsto \mathsf{t} \quad \mathsf{f}\ a\ \mathsf{f} \mapsto \mathsf{f}$$

- How does that change the rules?

Contraction:

$$\frac{(\mathsf{t}\ a\ \mathsf{f}) \vee (\mathsf{t}\ a\ \mathsf{f})}{(\mathsf{t} \vee \mathsf{t})\ a\ (\mathsf{f} \vee \mathsf{f})} \mapsto \frac{\bar{a} \vee \bar{a}}{\bar{a}}$$

But how?

Two more examples, identity and cut:

$$\frac{(\mathsf{f} \vee \mathsf{t})\; a\; (\mathsf{t} \vee \mathsf{f})}{(\mathsf{f}\; a\; \mathsf{t}) \vee (\mathsf{t}\; a\; \mathsf{f})} \mapsto \frac{\mathsf{t}}{a \vee \bar{a}} \qquad \text{and} \qquad \frac{(\mathsf{f}\; a\; \mathsf{t}) \wedge (\mathsf{t}\; a\; \mathsf{f})}{(\mathsf{f} \wedge \mathsf{t})\; a\; (\mathsf{t} \wedge \mathsf{f})} \mapsto \frac{a \wedge \bar{a}}{\mathsf{f}} \quad .$$

They are generated by the linear schemes:

$$\frac{(A \vee C)\; a\; (B \vee D)}{(A\; a\; B) \vee (C\; a\; D)} \qquad \text{and} \qquad \frac{(A\; a\; C) \wedge (B\; a\; D)}{(A \wedge B)\; a\; (C \wedge D)}$$

- Surprisingly, we are able to reduce disparate rules such as contraction, cut and identity into a unique rule scheme.

But how?

- Can we make proof systems for that?
  - Not in Gentzen formalisms.
  - Yes in Deep Inference.

- Deep Inference is necessary for complete proof systems with self-dual non-commutative connectives [5].

But how?

$$\mathsf{ai}{\downarrow}\ \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\ \frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{m}\ \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}{\downarrow}\ \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\ \frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\ \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\ \frac{S\{a\}}{S(a,a)}$$

Figure: SKS [1]

$$i{\downarrow}\ \frac{[A \vee B]\ a\ [C \vee D]}{\langle A\ a\ C\rangle \vee \langle B\ a\ D\rangle} \qquad i{\uparrow}\ \frac{\langle A\ a\ B\rangle \wedge \langle C\ a\ D\rangle}{(A \wedge C)\ a\ (B \wedge D)}$$

$$\frac{[A \vee B] \wedge [C \vee D]}{(A \wedge C) \vee [B \vee D]}\ s$$

$$\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}\ m$$

$$c{\downarrow}\ \frac{\langle A\ a\ B\rangle \vee \langle C\ a\ D\rangle}{[A \vee C]\ a\ [B \vee D]} \qquad c{\uparrow}\ \frac{(A \wedge B)\ a\ (C \wedge D)}{\langle A\ a\ C\rangle \wedge \langle B\ a\ D\rangle}$$

Figure: SAKS

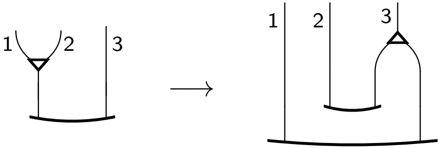
With subatomic logic

- We can generally study the interactions between rules.
- Some results:
  - We can represent a wide variety of systems with a single rule scheme, including CL and LL.
  - We provide a general cut-elimination theorem for a whole class of substructural logics.
  - In fact, we prove admissibility of a whole class of rules in a procedure of polynomial-time complexity.

We put it to use to generalise decomposition.

General decomposition

$$
ai\uparrow\,\frac{\boxed{ac\downarrow\,\dfrac{a^1 \vee a^2}{a}} \wedge \bar{a}^3}{\mathsf{f}}
\quad\longrightarrow\quad
=\,\frac{s\,\dfrac{s\,\dfrac{(a^1 \vee a^2) \wedge \boxed{ac\uparrow\,\dfrac{\bar{a}^3}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{ai\uparrow\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{ai\uparrow\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}}
$$

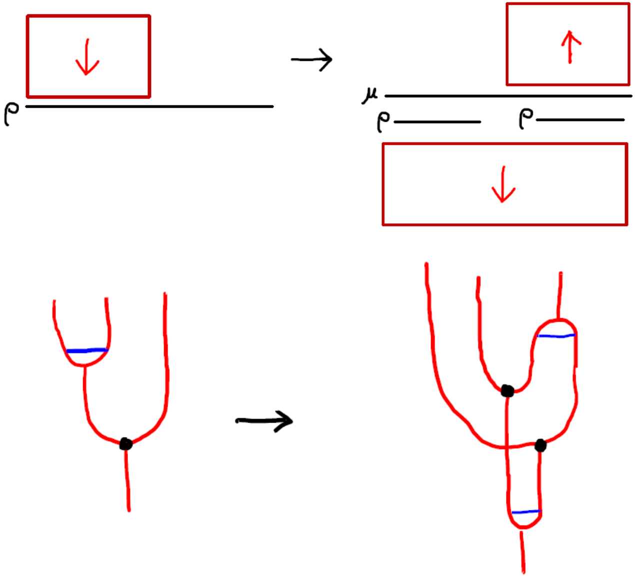


General decomposition

$$
i{\uparrow}\,\dfrac{\boxed{c{\downarrow}\,\dfrac{(\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{t})}{\mathsf{f}\ a\ \mathsf{t}}} \wedge \mathsf{t}\ a\ \mathsf{f}}{\mathsf{f}} \quad \longrightarrow
$$

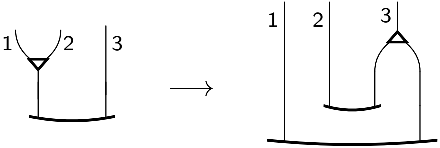
$$
\begin{array}{ll}
 & ((\mathsf{f}\ a\ \mathsf{t}) \vee (\mathsf{f}\ a\ \mathsf{t})) \wedge \boxed{c{\uparrow}\,\dfrac{\mathsf{t}\ a\ \mathsf{f}}{(\mathsf{t}\ a\ \mathsf{f}) \wedge (\mathsf{t}\ a\ \mathsf{f})}} \\
s & \overline{\quad ((\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{t}\ a\ \mathsf{f}) \wedge (\mathsf{t}\ a\ \mathsf{f})) \vee (\mathsf{f}\ a\ \mathsf{t}) \quad} \\
s & \overline{\quad i{\uparrow}\,\dfrac{(\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{t}\ a\ \mathsf{f})}{(\mathsf{f} \wedge \mathsf{t})\ a\ (\mathsf{t} \wedge \mathsf{f})} \vee i{\uparrow}\,\dfrac{(\mathsf{f}\ a\ \mathsf{t}) \wedge (\mathsf{t}\ a\ \mathsf{f})}{(\mathsf{f} \wedge \mathsf{t})\ a\ (\mathsf{t} \wedge \mathsf{f})} \quad} \\
c{\downarrow} & \overline{\quad ((\mathsf{f} \wedge \mathsf{t}) \vee (\mathsf{f} \wedge \mathsf{t}))\ a\ ((\mathsf{t} \wedge \mathsf{f}) \vee (\mathsf{t} \wedge \mathsf{f})) \quad}
\end{array}
$$

General decomposition



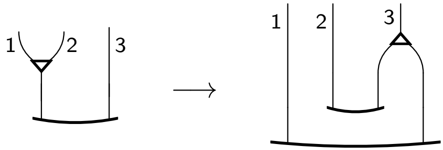
General decomposition

$$\text{ai}\uparrow \dfrac{\boxed{\text{ac}\downarrow \dfrac{a^1 \vee a^2}{a}} \wedge \bar{a}^3}{\mathsf{f}} \quad \longrightarrow \quad = \dfrac{s\dfrac{s\dfrac{(a^1 \vee a^2) \wedge \boxed{\text{ac}\uparrow \dfrac{\bar{a}^3}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{\text{ai}\uparrow \dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{\text{ai}\uparrow \dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}}$$



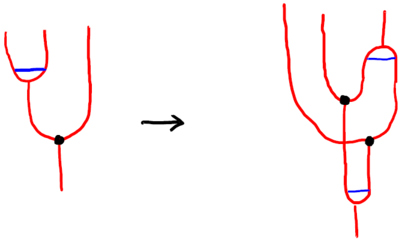
General decomposition

$$
ai\!\uparrow\,\dfrac{\boxed{ac\!\downarrow\,\dfrac{a^1 \vee a^2}{a}} \wedge \bar{a}^3}{\mathsf{f}}
\quad\longrightarrow\quad
=\,\dfrac{s\,\dfrac{s\,\dfrac{(a^1 \vee a^2) \wedge \boxed{ac\!\uparrow\,\dfrac{\bar{a}^3}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{ai\!\uparrow\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{ai\!\uparrow\,\dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}}
$$



General decomposition

$$ai\uparrow\ \dfrac{\boxed{ac\downarrow\ \dfrac{a^1 \vee a^2}{a}} \wedge \bar{a}^3}{\mathsf{f}} \quad \longrightarrow \quad {=}\ \dfrac{s\ \dfrac{s\ \dfrac{(a^1 \vee a^2) \wedge \boxed{ac\uparrow\ \dfrac{\bar{a}^3}{\bar{a} \wedge \bar{a}}}}{(a \wedge (\bar{a} \wedge \bar{a})) \vee a}}{\boxed{ai\uparrow\ \dfrac{a \wedge \bar{a}}{\mathsf{f}}} \vee \boxed{ai\uparrow\ \dfrac{a \wedge \bar{a}}{\mathsf{f}}}}}{\mathsf{f}}$$



General decomposition

- This reduction shape can be observed frequently.
- It can be generalised to provide reductions for all contractive rules.
- We can characterise those systems for which these reduction rules are sound.

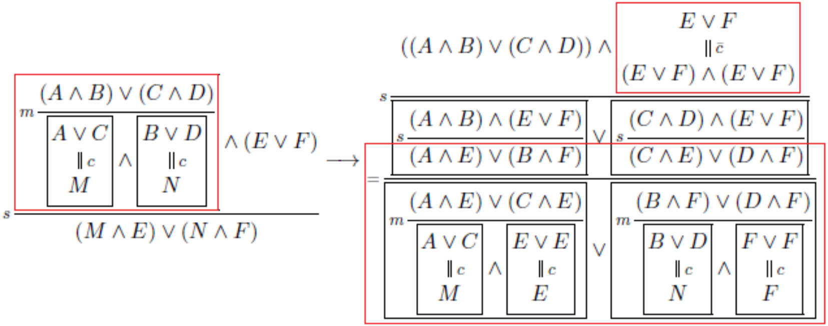
General decomposition

Theorem (Pending approval)

We can decompose derivations into an introductory phase followed by a contractive phase.

(In a certain class of systems including CL and MALL)

General decomposition

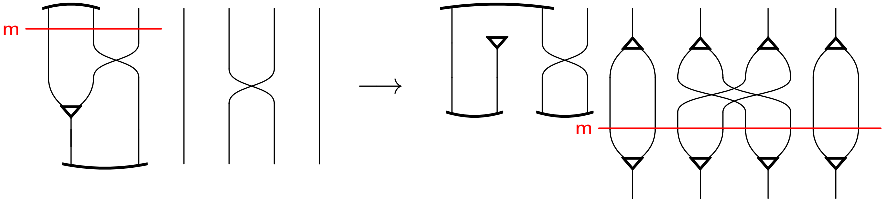


Cycle elimination

Theorem

Given a derivation with a cycle, there exists a cycle-free derivation with the same premiss and conclusion.

(In CL and MALL.)



Conclusions

- We observe a striking phenomenon: only one rule shape is enough to describe many different systems.
- We are able to observe that complexity comes from decomposition rather than from splitting.
- We wonder what role cycles play as a compression mechanism.
- We would like to use it as a stepping stone towards a geometrical formalism.

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