

A story of additives and multiplicatives

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Untangling cut-elimination

- ▶ In traditional cut-elimination procedures in Gentzen theory, we eliminate cut instances from proofs by moving upwards instances of the mix rule.

$$\frac{\vdash mA, \Gamma \quad \vdash n\bar{A}, \Delta}{\vdash \Gamma, \Delta}$$

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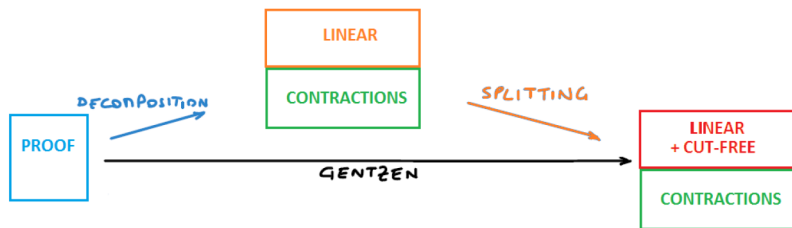
$$\frac{\vdash mA, \Gamma \quad \vdash n\bar{A}, \Delta}{\vdash \Gamma, \Delta}$$

- ▶ The presence of contraction makes for a jump to a higher complexity class.

Untangling cut-elimination

- ▶ Can we untangle cut and contraction and normalise on each of them **separately** and in a natural way?
- ▶ If so, what influence does the shape of rules have on normalisation?

Untangling cut- elimination



- ▶ **Decomposition** is the normalisation of contractions by permuting them to the bottom of proofs. It can increase the size of proofs exponentially.
- ▶ **Splitting** deals with cut-elimination in contraction-free systems. It does not generate meaningful complexity.

Locality

- ▶ It is essential to move away from the sequent calculus: it is always possible to build a valid sequent for which there is no sequent calculus proof where all the contractions are confined to the bottom (Brünnler, 2003).
- ▶ **Locality is fundamental.**

Why Deep Inference?

- ▶ Rules can be applied at any depth inside a formula.
- ▶ Rules can be made atomic and present great regularity.
- ▶ Atomic contractions can be permuted to the bottom of a derivation (Gundersen, 2009; Straßburger, 2003).

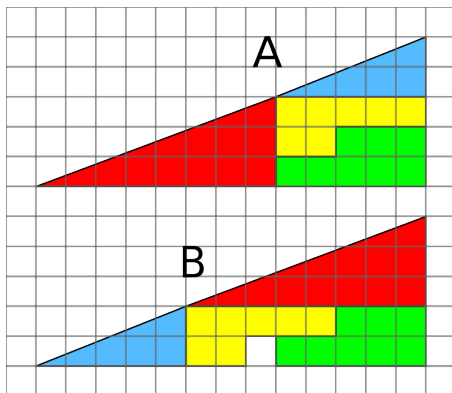
Splitting

- ▶ We find all the subproofs that are independent from each other above the multiplicative 'cut' connective.

$$\begin{array}{c} \begin{array}{c} \text{H}_1 \\ \text{H}_2 \end{array} \\ \frac{\begin{array}{c} \vdash A, \Phi \quad \vdash B, \Psi \end{array}}{\vdash A \otimes B, \Phi, \Psi} \\ \otimes \\ \Delta \\ \vdash F\{A \otimes B\}, \Gamma \end{array}$$

Splitting

- ▶ We put them back together in such a way that we obtain a proof with the same conclusion but without the cuts.



Splitting

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$$\begin{array}{c}
 \boxed{H_a \otimes \frac{1}{\bar{a} \wp a}} \otimes \frac{1}{\bar{a} \wp a} \otimes \boxed{H_{\bar{a}}} \\
 \parallel \qquad \qquad \qquad \parallel \\
 K_a \wp a \qquad \qquad \qquad \bar{a} \wp K_{\bar{a}} \\
 \hline
 K_a \wp \xrightarrow{a \otimes \bar{a}} \wp K_{\bar{a}} \\
 \perp
 \end{array}
 \xrightarrow{\text{splitting}}
 \begin{array}{c}
 H_a \otimes \frac{1}{\bar{a} \wp a} \otimes H_{\bar{a}} \\
 \hline
 \boxed{H_a \otimes \bar{a}} \wp \boxed{a \otimes H_{\bar{a}}} \\
 \parallel \qquad \qquad \qquad \parallel \\
 K_a \qquad \qquad \qquad K_{\bar{a}}
 \end{array}$$

Regularity

$\frac{1}{a \wp \bar{a}}$	$\frac{a \otimes \bar{a}}{\perp}$
$\otimes \downarrow \frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \wp D)}$	$\wp \uparrow \frac{(A \wp B) \otimes (C \otimes D)}{(A \otimes C) \wp (B \otimes D)}$
$\& \downarrow \frac{(A \wp B) \& (C \wp D)}{(A \& C) \wp (B \oplus D)}$	$\oplus \uparrow \frac{(A \oplus B) \otimes (C \& D)}{(A \otimes C) \oplus (B \otimes D)}$
$\oplus \downarrow \frac{(A \wp B) \oplus (C \wp D)}{(A \oplus C) \wp (B \oplus D)}$	$\& \uparrow \frac{(A \& B) \otimes (C \& D)}{(A \otimes C) \& (B \otimes D)}$
$\overset{m}{\frac{(A \& B) \oplus (C \& D)}{(A \oplus C) \& (B \oplus D)}}$	
$\overset{m_1}{\frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)}}$	$\overset{m_2}{\frac{(A \& B) \wp (C \& D)}{(A \wp C) \& (B \wp D)}}$
$\text{ac} \downarrow \frac{a \oplus a}{a}$	$\text{ac} \uparrow \frac{a}{a \& a}$
$\text{aw} \downarrow \frac{0}{a}$	$\text{aw} \uparrow \frac{a}{\top}$

Figure: System SLLS (Straßburger, 2003)

Regularity

$\frac{1}{a \wp \bar{a}}$	$\frac{a \otimes \bar{a}}{\perp}$
$\otimes\downarrow \frac{(A \wp B) \otimes (C \wp D)}{(A \otimes C) \wp (B \wp D)}$	$\wp\uparrow \frac{(A \wp B) \otimes (C \otimes D)}{(A \otimes C) \wp (B \otimes D)}$
$\&\downarrow \frac{(A \wp B) \& (C \wp D)}{(A \& C) \wp (B \oplus D)}$	$\oplus\uparrow \frac{(A \oplus B) \otimes (C \& D)}{(A \otimes C) \oplus (B \otimes D)}$
$\oplus\downarrow \frac{(A \wp B) \oplus (C \wp D)}{(A \oplus C) \wp (B \oplus D)}$	$\&\uparrow \frac{(A \& B) \otimes (C \& D)}{(A \otimes C) \& (B \otimes D)}$

← splittable rules

+

$\text{m} \frac{(A \& B) \oplus (C \& D)}{(A \oplus C) \& (B \oplus D)}$	
$\text{m}_1 \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)}$	$\text{m}_2 \frac{(A \& B) \wp (C \& D)}{(A \wp C) \& (B \wp D)}$
$\text{ac}\downarrow \frac{a \oplus a}{a}$	$\text{ac}\uparrow \frac{a}{a \& a}$
$\text{aw}\downarrow \frac{0}{a}$	$\text{aw}\uparrow \frac{a}{\perp}$

← contraction rules

Figure: System SLLS

Regularity

$\frac{1}{a \wp \bar{a}}$	$\frac{a \otimes \bar{a}}{\perp}$
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+

$\frac{(A \& B) \oplus (C \& D)}{(A \oplus C) \& (B \oplus D)}$	
$m_1 \frac{(A \otimes B) \oplus (C \otimes D)}{(A \oplus C) \otimes (B \oplus D)}$	$m_2 \frac{(A \& B) \wp (C \& D)}{(A \wp C) \& (B \wp D)}$
$ac\downarrow \frac{a \oplus a}{a}$	$ac\uparrow \frac{a}{a \& a}$
$aw\downarrow \frac{0}{a}$	$aw\uparrow \frac{a}{\perp}$

← contraction rules

Figure: System SLLS

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← contraction rules

Figure: System SLLS

Regularity

$$\boxed{\begin{array}{cc} \text{ai}\downarrow \frac{t}{a \vee \bar{a}} & \text{ai}\uparrow \frac{a \wedge \bar{a}}{f} \\ \wedge\downarrow \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} & \vee\uparrow \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \end{array}}$$

+

$$\boxed{\begin{array}{cc} \text{m} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} & \\ \text{c}\downarrow \frac{a \vee a}{a} & \text{c}\uparrow \frac{a}{a \wedge a} \\ \text{aw}\downarrow \frac{f}{a} & \text{aw}\uparrow \frac{a}{t} \end{array}}$$

Figure: System cSKS

← splittable rules

The rules on the right are **admissible**.

← contraction rules

The rules on the left will permute down and the rules on the right will permute up.

Splitting

Definition

A system S is *splittable* when:

1. There are dual distinguished connectives \times with unit 1 and $+$ with unit 0.
2. S is uniquely composed of the rules

$$\text{ai}\downarrow \frac{1}{a + \bar{a}} \quad \text{and} \quad \text{ai}\uparrow \frac{a \times \bar{a}}{0} ,$$

together with rules

$$\alpha\downarrow \frac{(A + B) \alpha (C + D)}{(A \alpha C) + (B \check{\alpha} D)} \quad \text{and} \quad \alpha\uparrow \frac{(A \hat{\alpha} B) \times (C \alpha D)}{(A \times C) \alpha (B \times D)}$$

for every connective α .

3. For every unit u , $u + \bar{u} = 1$.
4. For every connective α , $1 \hat{\alpha} 1 = 1$.

Subatomic logic

$$\frac{(\perp \wp 1) a (1 \wp \perp)}{(\perp a 1) \wp (1 a \perp)} \mapsto \frac{1}{a \wp \bar{a}}$$

$$\frac{(\perp a 1) \otimes (1 a \perp)}{(\perp \otimes 1) a (1 \otimes \perp)} \mapsto \frac{a \otimes \bar{a}}{\perp}$$

They are generated by the linear schemes:

$$\frac{(A \wp B) a (C \wp D)}{(A a C) \wp (B a D)} \quad \text{and} \quad \frac{(A a B) \otimes (C a D)}{(A \otimes C) a (B \otimes D)}$$

- ▶ We are able to reduce disparate rules such as contraction, cut and identity into a unique rule scheme.

Splitting

Theorem

Let S be a splittable system. For every proof

$$\phi \Vdash_S A$$

there is a proof

$$\psi \Vdash_S \{a_i \uparrow, \alpha \uparrow\}$$

linear on the size of ϕ , and where ψ can be obtained from ϕ in a procedure of polynomial-time complexity.

Splittable systems

- ▶ A whole class of substructural logics are splittable.
- ▶ Including logics that support self-dual non commutative connectives, such as BV.
- ▶ MLL is splittable, the linear fragment of CL is splittable...
- ▶ A whole family of rules is admissible.

Decomposition

- ▶ In several deep inference systems, we know that we can permute **atomic** contractions to the bottom of proofs through local reductions.
- ▶ We want to restrict all contraction rules to the bottom of proofs.

Decomposition

- ▶ We will present a rewriting system to permute contractions below any other rule and cocontractions above any other rule.
- ▶ We will sketch a proof of the termination of this rewriting system.

Decomposable systems

'Definition'

A system SD is *decomposable* if:

it is made-up of a splittable system plus contraction and weakening (either atomic contraction and medials, or unbounded contraction).

Decomposable systems

Definition

A system SD is *decomposable* if:

1. There are dual distinguished relations \sqcup with unit w and \sqcap with unit \bar{w} .
2. SD is composed of a splittable system S with, together with the rules

$$\begin{array}{ccc} c\downarrow \frac{A \sqcup A}{A} & \text{and} & c\uparrow \frac{A}{A \sqcap A} \\ aw\downarrow \frac{w}{a} & \text{and} & aw\uparrow \frac{a}{\bar{w}} \end{array} .$$

3. For every unit u , $u \sqcup u = u = u \sqcap u$.
4. For every connective α , $w \alpha w = w$ and $\bar{w} \alpha \bar{w} = \bar{w}$.

Decomposition

- ▶ Splittable systems have only two types of rules

$$\alpha\downarrow \frac{(A + B) \alpha (C + D)}{(A \alpha C) + (B \check{\alpha} D)} \quad \text{and} \quad \alpha\uparrow \frac{(A \hat{\alpha} B) \times (C \alpha D)}{(A \times C) \alpha (B \times D)} .$$

- ▶ Therefore we only need to study 3 interactions: contractions with down-rules, contractions with up-rules, and contractions with cocontractions.

Trivial reduction

► Reduction $t \downarrow$:

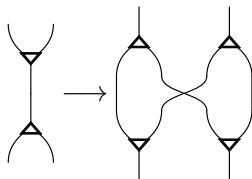
$$\rho \frac{\boxed{A \sqcup A}}{A} \quad \longrightarrow \quad c \downarrow \frac{\boxed{\rho \frac{A}{C}} \sqcup \boxed{\rho \frac{A}{C}}}{C}$$

Trivial reduction

- ▶ Reduction $t \downarrow$:

$$c\uparrow \frac{\boxed{\frac{A \sqcup A}{A}}}{A \sqcap A} \longrightarrow c\downarrow \frac{\frac{c\uparrow \frac{A}{A \sqcap A} \sqcup c\uparrow \frac{A}{A \sqcap A}}{A \sqcap A}}$$

- ▶ This can cause an exponential explosion in the size of the proof.



Permuting past up-rules

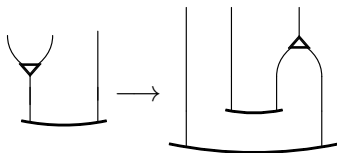
► Reduction $u \downarrow$:

$$\begin{array}{c}
 \boxed{\text{c}\downarrow \frac{(A_1 \alpha A_2) \sqcup (A_1 \alpha A_2)}{A_1 \alpha A_2}} \times (C \hat{\alpha} D) \xrightarrow{\alpha \uparrow} \\
 (A_1 \times C) \alpha (A_2 \times D) \\
 \\
 ((A_1 \alpha A_2) \sqcup (A_1 \alpha A_2)) \times \boxed{\text{c}\uparrow \frac{C \hat{\alpha} D}{(C \hat{\alpha} D) \sqcap (C \hat{\alpha} D)}} \\
 \sqcup \uparrow \\
 \boxed{\alpha \uparrow \frac{(A_1 \alpha A_2) \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)}} \sqcup \boxed{\alpha \uparrow \frac{(A_1 \alpha A_2) \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)}} \\
 \text{c}\downarrow \hline
 (A_1 \times C) \alpha (A_2 \times D)
 \end{array}$$

Permuting past up-rules

- Reduction $u_a \downarrow$:

$$\text{ai}\uparrow \frac{\boxed{\text{c}\downarrow \frac{a \vee a}{a}}}{f} \wedge \bar{a} \longrightarrow \vee \uparrow \frac{(a \vee a) \wedge \boxed{\text{c}\uparrow \frac{\bar{a}}{(\bar{a} \wedge \bar{a})}}}{\boxed{\text{ai}\uparrow \frac{a \wedge \bar{a}}{f}} \vee \boxed{\text{ai}\uparrow \frac{a \wedge \bar{a}}{f}}}$$

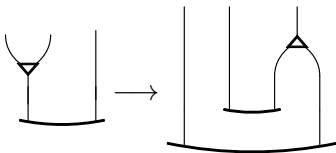


Permuting past up-rules

► Reduction $u \downarrow$:

$$\alpha_j \uparrow \frac{\boxed{c \downarrow \frac{(A_1 \alpha A_2) \sqcup (A_1 \alpha A_2)}{A_1 \alpha A_2}} \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)} \longrightarrow$$

$$\begin{array}{c} ((A_1 \alpha A_2) \sqcup (A_1 \alpha A_2)) \times \boxed{c \uparrow \frac{C \hat{\alpha} D}{(C \hat{\alpha} D) \sqcap (C \hat{\alpha} D)}} \\ \sqcup_j \uparrow \frac{\alpha_j \uparrow \frac{(A_1 \alpha A_2) \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)}}{\boxed{\alpha_j \uparrow \frac{(A_1 \alpha A_2) \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)}}} \sqcup \frac{\alpha_j \uparrow \frac{(A_1 \alpha A_2) \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)}}{\boxed{\alpha_j \uparrow \frac{(A_1 \alpha A_2) \times (C \hat{\alpha} D)}{(A_1 \times C) \alpha (A_2 \times D)}}} \\ c \downarrow \frac{\quad}{(A_1 \times C) \alpha (A_2 \times D)} \end{array}$$



Permuting past down-rules

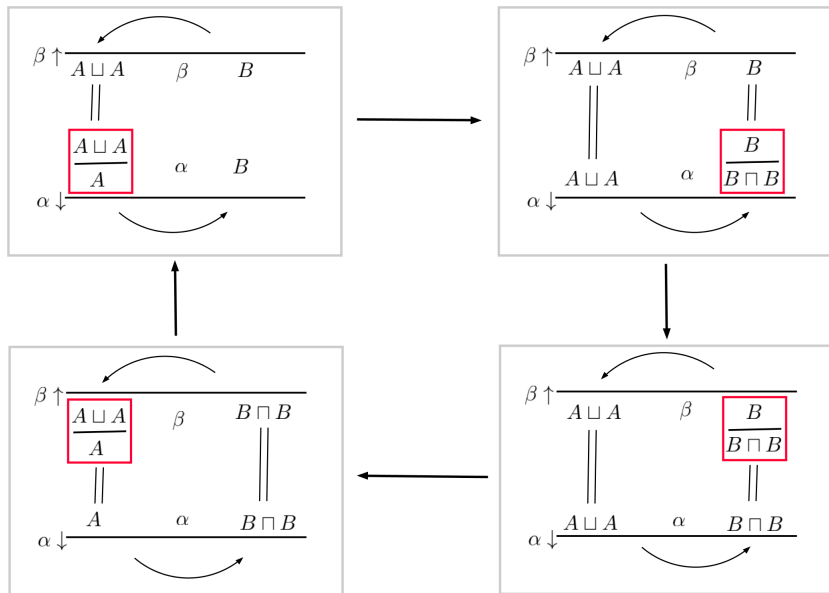
► Reduction $r\downarrow$:

$$\begin{array}{c}
 \boxed{\text{c}\downarrow \frac{(A_1 + A_2) \sqcup (A_1 + A_2)}{A_1 + A_2}} \alpha (D + E) \longrightarrow \\
 \alpha\downarrow \frac{\quad}{(A_1 \alpha D) + (A_2 \check{\alpha} E)} \\
 \\
 \boxed{\sqcup\downarrow \frac{(A_1 + A_2) \sqcup (A_1 + A_2)}{(A_1 \sqcup A_1) + (A_2 \sqcup A_2)}} \alpha (D + E) \\
 \alpha\downarrow \frac{\quad}{\left(\boxed{\text{c}\downarrow \frac{A_1 \sqcup A_1}{A_1}} \alpha D \right) + \left(\boxed{\text{c}\downarrow \frac{A_2 \sqcup A_2}{A_2}} \check{\alpha} E \right)}
 \end{array}$$

Termination (the issues)

- ▶ The rewriting system is non-confluent.
- ▶ When contractions are permuted downwards they may be duplicated.
- ▶ When we permute contractions we create cocontractions, and vice-versa.
- ▶ There may be cycles.

Cycles



Cycles

$$\begin{array}{c}
 \text{ai}\downarrow \frac{t}{a \vee \bar{a}} \wedge (b \wedge c) \\
 \frac{2s}{(a \wedge c) \vee (b \wedge \bar{a})} \wedge (d \wedge e) \\
 \text{m} \frac{[a \vee b] \wedge [c \vee \bar{d}]}{=} \\
 \frac{s}{a \vee (b \wedge d)} \wedge \frac{s}{\bar{a} \vee (c \wedge e)} \\
 \frac{2s}{a \wedge \bar{a}} \vee [(b \wedge d) \vee (c \wedge e)] \\
 \text{ai}\uparrow \frac{f}{\quad}
 \end{array}$$

- By permuting medials down, we can remove cycles **locally** and without resorting to cut-elimination.

Decomposition

Theorem

In a decomposable system, there is a reduction strategy so that the rewriting system terminates.

Equivalently, every proof can be rewritten as a proof where all instances of contraction are at the bottom of the proof (and there are no instances of cocontraction).

Decomposition

1. We permute all contractions downwards, always starting from the lowest. When a contraction faces a cocontraction, we duplicate the cocontraction. We show that this procedure terminates.
2. We permute all cocontraction upwards, with the dual strategy.
3. We iterate between steps 1 and 2, and show that the iteration terminates.

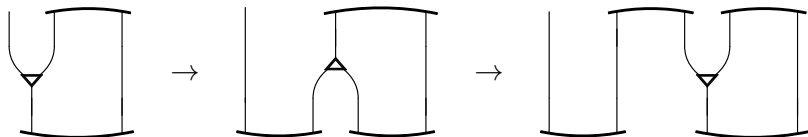
It is a generalization of (Straßburger, 2003).

Conclusions

- ▶ We can give a simple classification of rules in terms of their behaviour in normalisation.
- ▶ Non-confluence and complexity creation are restricted to the decomposition phase of cut-elimination.
- ▶ We give a uniform treatment for many existing logics
- ▶ We can use these results to design systems with guaranteed modular cut-elimination.
- ▶ We can control complexity by following atoms.

Open problems

- ▶ Cycles are inevitable as long as there are contractions.
- ▶ They are not a DI construct (Carbone, 1997)
- ▶ We have two procedures to eliminate them, and both involve arbitrarily big duplications. How much do they compress proofs?



Open problems

- Complexity is created by permuting contractions through associativity instances i.e. complexity is created in inference steps that have no logical content in terms of deduction.

Two proofs differing only through redundant instances of associativity will have different decomposed forms, of different sizes.

$$\begin{array}{c}
 \boxed{\begin{array}{c}
 \boxed{\begin{array}{c}
 \text{ai} \downarrow \frac{t}{a \vee \bar{a}} \wedge (b \wedge c) \\
 (a \wedge c) \vee (b \wedge \bar{a}) \\
 m \\
 [a \vee b] \wedge [c \vee \bar{a}]
 \end{array}} \vee (d \wedge e) \\
 2s \\
 \frac{m}{[[a \vee b] \vee d] \wedge [[c \vee \bar{a}] \vee e]} \wedge g
 \end{array}} \\
 = \\
 \frac{m}{[[a \vee b] \vee d] \wedge [[c \vee \bar{a}] \vee e] \wedge g} \\
 = \\
 \boxed{\begin{array}{c}
 \frac{2s}{[a \vee b] \wedge [c \vee \bar{a}]} \\
 \frac{2s}{\left[\frac{\text{ai} \uparrow \frac{a \wedge \bar{a}}{f} \vee b \right] \vee c} \vee [d \vee e]} \wedge g
 \end{array}}
 \end{array}
 \rightarrow
 \begin{array}{c}
 \frac{2s}{\left[\frac{b \wedge c}{t} \vee \frac{d \wedge e}{t} \right] \wedge \frac{g}{g \wedge g}} \\
 \frac{m}{(c \wedge g) \vee (e \wedge g)} \\
 \frac{m}{\left[\frac{c}{f} \vee \frac{e}{d} \right] \wedge \frac{g \vee g}{g}}
 \end{array}$$

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